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LETTER TO THE EDITOR

A deformed quantum SU(2) superalgebra

B Aneva

Institute of Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 1784 Sofia, Bulgaria

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Abstract. A quantum q-deformation of the SU(2) superalgebra is obtained which contains as an even part the quantum $\mathrm{SU}_q(2)$ algebra. The finite-dimensional irreducible representations of the deformed quantum superalgebra are found.

Recently there has been great interest in the study of new types of algebraic structures, loosely called quantum groups. In precise mathematical language these objects have been categorized in terms of Hopf algebras by Drinfeld [1] and have been intensively developed by Jimbo [2-4] and Woronowicz [5, 6]. More recently a realization of the quantum groups $\mathrm{SU}_q(2)$ and $\mathrm{SU}_q(1,1)$ have been constructed [7, 8] using a q-generalization of the Jordan-Schwinger approach to group representation theory by means of a q-analogue of the boson operator calculus.

The boson realization method is a very powerful and elegant method for the study of group representations. It has been successfully applied to construct representations of Lie algebras [9], Lie superalgebras [10] and loop algebras [11]. The quantum analogue of the simplest superalgebra generated by three elements has been considered in [12]. In this letter we apply the q-generalization of the Jordan-Schwinger method to obtain a q-deformation of the SU(2) superalgebra which contains the quantum $\mathrm{SU}_q(2)$ algebra as an even part. In the limit $q\to 1$ the deformed quantum SU(2) superalgebra contracts to the SU(2) Lie superalgebra.

We briefly sketch the SU(2) superalgebra and its representations. The even part is the Lie algebra of SU(2) generated by J_3, $J_\pm = J_1 \pm \mathrm{i}J_2$ with commutation relations

$$[J_3, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2J_3. \tag{1}$$

The odd generators Q_α ($\alpha = 1, 2$) form a SU(2) spinor

$$[J_k, Q_\alpha] = -\tfrac{1}{2}(\sigma_k Q)_\alpha \qquad k = 1, 2, 3 \tag{2}$$

with σ_k denoting the Pauli matrices. Q_α satisfy

$$\{Q_\alpha, Q_\beta\} = 2(\sigma_k\varepsilon)_{\alpha\beta}J_k \qquad \varepsilon_{ij} = -\varepsilon_{ji}. \tag{3}$$

The finite-dimensional irreducible representations with the highest-weight vector are labelled by a quantum number $j = \frac{1}{2}$ or $j = 1$ where 1 is a natural number. With respect to the even subalgebra the j-representation is decomposed into a direct sum of two irreducible representations corresponding to spin j and $j - \frac{1}{2}$.

In order to obtain a q-deformed SU(2) superalgebra we shall follow the procedure of applying the Jordan-Schwinger approach to Lie superalgebras. We start with a pair of q-analogues of boson creation and annihilation operators a_i and a_i^+ and the corresponding number operators N_i $(i=1,2)$, which obey (for equal i) the relations:

$$a_i a_i^+ - q^{1/2} a_i^+ a_i = q^{-N_i/2} \tag{4}$$

$$a_i a_i^+ - (1/q)^{1/2} a_i^+ a_i = (1/q)^{-N_i/2} \tag{5}$$

$$[N_i, a_i] = -a_i \qquad [N_i, a_i^+] = a_i^+.$$

We suppose now that $a_i^+ = \varepsilon_{ij} a_j$, $\varepsilon_{ij} = -\varepsilon_{ji}$, $\varepsilon_{12} = 1$ and analyse the above relations with this supposition. We have

$$a_1 a_2 - q^{1/2} a_2 a_1 = q^{-N_1/2} \tag{4a}$$

$$-a_2 a_1 + q^{1/2} a_1 a_2 = q^{-N_2/2} \tag{4b}$$

and correspondingly for $q \to 1/q$

$$a_1 a_2 - (1/q)^{1/2} a_2 a_1 = (1/q)^{-N_1/2} \tag{5a}$$

$$-a_2 a_1 + (1/q)^{1/2} a_1 a_2 = (1/q)^{-N_2/2}. \tag{5b}$$

Comparing (4a) with (5b) and (4b) with (5a) we conclude that if the relations (4) and (5) are to be consistent we must have

$$-N_1 = N_2 + 1 \qquad \text{and} \qquad -N_2 = N_1 + 1 \tag{6}$$

We solve now (4) or (5) with respect to $a_1 a_2$ and $a_2 a_1$ and find

$$a_1 a_2 = -[N_2] \qquad a_2 a_1 = [N_1] \tag{7}$$

where $[A]$ is defined as

$$[A] = \frac{q^{A/2} - q^{-A/2}}{q^{1/2} - q^{-1/2}} \qquad q \in R. \tag{8}$$

The interpretation of the relation $-N_1 + -N_2 = 1$ between the two N-operators is simply that only one of them may have the meaning of a number operator with respect to which a_1 and a_2 are creation and annihilation operators respectively:

$$\begin{matrix} [N_1, a_1] = -a_1 \\ [N_1, a_2] = a_2 \end{matrix} \qquad \text{or} \qquad \begin{matrix} [N_2, a_1] = a_1 \\ [N_2, a_2] = -a_2. \end{matrix} \tag{9}$$

We argue that given the quantum group SU_q (2), which is generated algebraically by the operators J_+, J_- and J_3 obeying the Lie bracket (commutator) relations

$$[J_3, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = [2J_3] \tag{10}$$

and given the two operators $a_{1,2}$, defined by the relations (7) and (9), provided $-N_1 - N_2 = 1$, we can construct a deformed quantum superalgebra the even part of which is the algebra (10).

From (7) we form the anticommutator

$$\{a_1, a_2\} = [N_1] + [N_1 + 1]. \tag{11}$$

We postulate further

$$\begin{aligned} &\{a_1, a_1\} = g_-(N_1, N_2, q) J_- \\ & \qquad\qquad\qquad\qquad\qquad J_3 = \tfrac{1}{2}(N_1 - N_2). \\ &\{a_2, a_2\} = g_+(N_1, N_2, q) J_+ \end{aligned} \tag{12}$$

Due to the relation between N_1 and N_2 we have to choose for J_3 one of the expressions

$$J_3 = N_1 + \tfrac{1}{2} \qquad \text{or} \qquad J_3 = -N_2 - \tfrac{1}{2}.$$

In both cases the anticommutator (11) has the same form in terms of the operator J_3:

$$\{a_1, a_2\} = [J_3 + \tfrac{1}{2}] + [J_3 - \tfrac{1}{2}]. \tag{13}$$

We observe that due to (9) the operators a_1 and a_2 satisfy

$$[J_3, a_1] = -\tfrac{1}{2}a_1 \qquad [J_3, a_2] = \tfrac{1}{2}a_2. \tag{14}$$

Simple algebraic manipulations show that the operators $J_\pm$, J_3, defined by (12), satisfy the $SU_q(2)$ Lie algebra (10) provided the functions $g_\pm$ have the form

$$g_\pm(q, N_1, N_2) = \pm f(q) = \pm 2(q^{1/2} + q^{-1/2})^{1/2}.$$

To verify this result we have only used (9) and (7). Hence

$$\{a_1, a_1\} = -2(q^{1/2} + q^{-1/2})^{1/2} J_- \qquad \{a_2, a_2\} = 2(q^{1/2} + q^{-1/2})^{1/2} J_+. \tag{15}$$

To close the superalgebra we need further determine the commutator $[J_\pm, a_i]$ with $i = 1, 2$. From (4), (5) and the expressions (12) for $J_\pm$ in terms of $a_{1,2}$ it follows, with $J_3 = N_1 + \frac{1}{2}$:

$$\begin{aligned}
&[J_+, a_1] = -(q^{1/2} + q^{-1/2})^{-1/2}(q^{N_1/2} + q^{-N_1/2})a_2 \\
&[J_+, a_2] = 0 \qquad [J_-, a_1] = 0 \\
&[J_-, a_2] = -(q^{1/2} + q^{-1/2})^{-1/2}(q^{(N_1+1)/2} + q^{-(N_1+1)/2})a_1.
\end{aligned} \tag{16}$$

We have thus obtained a q-deformation of the SU(2) superalgebra, defined by the relations (10); (14), (16) and (13), (15). It is an infinite-dimensional associative superalgebra, a deformation of the universal enveloping of the SU(2) Lie superalgebra. It includes as an even part the quantum $SU_q(2)$ algebra. In the limit $q \to 1$ the deformed superalgebra contracts to the SU(2) Lie superalgebra.

The action of the quantum superalgebra generators in the tensor product of two representations is dfiened by the coproduct Δ

$$\begin{aligned}
&\Delta(J_3) = J_3 \otimes I + I \otimes J_3 \\
&\Delta(J_\pm) = J_\pm \otimes q^{J_3/2} + q^{-J_3/2} \otimes J_\pm \\
&\Delta(a_{1,2}) = a_{1,2} \otimes q^{J_3/4} + q^{-J_3/4} \otimes a_{1,2}
\end{aligned} \tag{17}$$

which amounts to a Hopf algebra structure for the quantum SU(2) superalgebra.

The ansatz $a_i^+ = \varepsilon_{ij} a_j$ exploited so far is a grade-adjoint operation on the quantum superalgebra defined as:

$$\begin{aligned}
&J_3^+ = J_3 \qquad J_\pm^+ = J_\pm \qquad a_{1,2}^+ = \pm a_{2,1} \\
&(AB)^+ = (-1)^{p(A)p(B)} B^+ A^+ \qquad (A^+)^+ = (-1)^{p(A)} A
\end{aligned} \tag{18}$$

where $p(A) = 0, 1$ is the degree of the homogeneous element A of the superalgebra.

We proceed further to construct a representation of the q-deformed quantum SU(2) superalgebra. As a superalgebra obeying the graded Lie brackets (10) and (13)–(16) it has representations with the same structure as in the classical case. We are interested in finite-dimensional irreducible representations. For this purpose we have to choose a finite-dimensional irrepresentation of the even subalgebra and then construct a representation of the superalgebra by the action of the odd generators. Given a representation $|j_0, m\rangle$ by the action of the odd generators. Given a representation $|j_0, m\rangle$ of $SU_q(2)$ (according to Jimbo a finite-dimensional irreducible representation of $SU_q(2)$ is labelled by a quantum number j, taking integer and half-integer values and acts in a Hilbert space V_j with dimension $2j+1$) then the representation space of the quantum superalgebra is spanned by the components $|j_0, m\rangle$, $a_1|j_0, m\rangle$, $a_2|j_0, m\rangle$ and $a_{[1}a_{2]}|j_0, m\rangle$. It contains two types of j-multiplets $|j, m\rangle$ and $|j-\frac{1}{2}, m\rangle$ belonging to spin j and $j-\frac{1}{2}$, with j integer or half-integer. The j-represenation admits an even invariant bilinear form $\langle|\rangle$ with respect to which the two spin j-multiplets are orthogonal. With a convenient normalization of the basis vectors the j-representation of the deformed SU(2) quantum superalgebra is defined as:

$$
\begin{aligned}
&J_3|j, m\rangle = 2m|j, m\rangle \qquad J_3|j-\tfrac{1}{2}, m\rangle = 2m|j-\tfrac{1}{2}, m\rangle\\
&J_\pm|j, m\rangle = (q^{1/2}+q^{-1/2})^{-1/2}([2j\mp 2m][2j\pm 2m+2])^{1/2}|j, m\pm 1\rangle\\
&J_\pm|j-\tfrac{1}{2}, m\rangle = (q^{1/2}+q^{-1/2})^{-1/2}([2j-1\mp 2m][2j+1\pm 2m])^{1/2}|j-\tfrac{1}{2}, m\pm 1\rangle\\
&a_1|j, m\rangle = [j+m]^{1/2}(q^{(j-m+1)/2}+q^{-(j-m+1)/2})^{1/2}|j-\tfrac{1}{2}, m-\tfrac{1}{2}\rangle\\
&a_1|j-\tfrac{1}{2}, m\rangle = -[j-m+\tfrac{1}{2}]^{1/2}(q^{(j+m+1/2)/2}+q^{-(j+m+1/2)/2})^{1/2}|j, m-\tfrac{1}{2}\rangle\\
&a_2|j, m\rangle = [j-m]^{1/2}(q^{(j+m+1)/2}+q^{-(j+m+1)/2})^{1/2}|j-\tfrac{1}{2}, m+\tfrac{1}{2}\rangle\\
&a_2|j-\tfrac{1}{2}, m\rangle = [j+m+\tfrac{1}{2}]^{1/2}(q^{(j-m+1/2)/2}+q^{-(j-m+1/2)/2})^{1/2}|j, m+\tfrac{1}{2}\rangle
\end{aligned}
\tag{19}
$$

where $m=-j, \ldots, j$.

In deriving (19) we made use of the fact that the representation is grade-adjoint with respect to the norm $\langle|\rangle$.

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